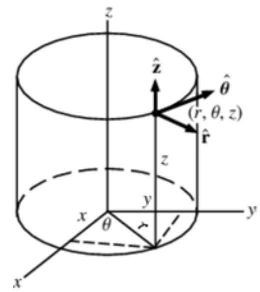
Cylindrical Coordinates



Cylindrical coordinates are a generalization of two-dimensional polar coordinates to three dimensions by superposing a height (z) axis. Unfortunately, there are a number of different notations used for the other two coordinates. Either r or ρ is used to refer to the radial coordinate and either ϕ or θ to the azimuthal coordinates. Arfken (1985), for instance, uses (ρ, ϕ, z), while Beyer (1987) uses (r, θ, z). In this work, the notation (r, θ, z) is used.

The following table summarizes notational conventions used by a number of authors.

(radial, azimuthal, vertical)	reference
(r, θ, z)	this work, Beyer (1987, p. 212)
(`Rr`, `Ttheta`, `Zz`)	`SetCoordinates[Cylindrical]` in the *Mathematica* package ``VectorAnalysis` ``
(ρ, ϕ, z)	Arfken (1985, p. 95)
(r, ψ, z)	Moon and Spencer (1988, p. 12)
(r', φ, z)	Korn and Korn (1968, p. 60)
(ξ_1, ξ_2, ξ_3)	Morse and Feshbach (1953)

In terms of the Cartesian coordinates (x, y, z),

$$r = \sqrt{x^2 + y^2} \qquad (1)$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) \qquad (2)$$

$$z = z, \qquad (3)$$

where $r \in [0, \infty)$, $\theta \in [0, 2\pi)$, $z \in (-\infty, \infty)$, and the inverse tangent must be suitably defined to take the correct quadrant of (x, y) into account.

In terms of x, y, and z

$$x = r\cos\theta \qquad (4)$$

$$y = r\sin\theta \qquad (5)$$

$$z = z. \qquad (6)$$

Note that Morse and Feshbach (1953) define the cylindrical coordinates by

$$x = \xi_1\,\xi_2 \qquad (7)$$

$$y = \xi_1\sqrt{1-\xi_2{}^2} \qquad (8)$$

$$z = \xi_3, \qquad (9)$$

where $\xi_1 = r$ and $\xi_2 = \cos\theta$.

The metric elements of the cylindrical coordinates are

$$g_{rr} = 1 \qquad (10)$$

$$g_{\theta\theta} = r^2 \qquad (11)$$

$$g_{zz} = 1, \qquad (12)$$

so the scale factors are

$$g_r = 1 \qquad (13)$$

$$g_\theta = r \qquad (14)$$

$$g_z = 1. \qquad (15)$$

The line element is

$$d\mathbf{s} = d\,r\,\hat{\mathbf{r}} + r\,d\,\theta\,\hat{\boldsymbol{\theta}} + d\,z\,\hat{\mathbf{z}}, \qquad (16)$$

and the volume element is

$$dV = r\,d\,r\,d\,\theta\,d\,z. \qquad (17)$$

The Jacobian is

$$\left|\frac{\partial(x, y, z)}{\partial(r, \theta, z)}\right| = r. \quad (18)$$

A Cartesian vector is given in cylindrical coordinates by

$$\mathbf{r} = \begin{bmatrix} r\cos\theta \\ r\sin\theta \\ z \end{bmatrix}. \quad (19)$$

To find the unit vectors,

$$\hat{\mathbf{r}} \equiv \frac{\frac{d\mathbf{r}}{dr}}{\left|\frac{d\mathbf{r}}{dr}\right|} = \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix} \quad (20)$$

$$\hat{\boldsymbol{\theta}} \equiv \frac{\frac{d\mathbf{r}}{d\theta}}{\left|\frac{d\mathbf{r}}{d\theta}\right|} = \begin{bmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{bmatrix} \quad (21)$$

$$\hat{\mathbf{z}} \equiv \frac{\frac{d\mathbf{r}}{dz}}{\left|\frac{d\mathbf{r}}{dz}\right|} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \quad (22)$$

Derivatives of unit vectors with respect to the coordinates are

$$\frac{\partial\hat{\mathbf{r}}}{\partial r} = \mathbf{0} \quad (23)$$

$$\frac{\partial\hat{\mathbf{r}}}{\partial\theta} = \hat{\boldsymbol{\theta}} \quad (24)$$

$$\frac{\partial\hat{\mathbf{r}}}{\partial z} = \mathbf{0} \quad (25)$$

$$\frac{\partial\hat{\boldsymbol{\theta}}}{\partial r} = \mathbf{0} \quad (26)$$

$$\frac{\partial\hat{\boldsymbol{\theta}}}{\partial\theta} = -\hat{\mathbf{r}} \quad (27)$$

$$\frac{\partial\hat{\boldsymbol{\theta}}}{\partial z} = \mathbf{0} \quad (28)$$

$$\frac{\partial\hat{\mathbf{z}}}{\partial r} = \mathbf{0} \quad (29)$$

$$\frac{\partial\hat{\mathbf{z}}}{\partial\theta} = \mathbf{0} \quad (30)$$

$$\frac{\partial\hat{\mathbf{z}}}{\partial z} = \mathbf{0}. \quad (31)$$

The gradient operator in cylindrical coordinates is given by

$$\nabla \equiv \hat{\mathbf{r}}\frac{\partial}{\partial r} + \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{\mathbf{z}}\frac{\partial}{\partial z}, \quad (32)$$

so the gradient components become

$$\nabla_r\hat{\mathbf{r}} = \mathbf{0} \quad (33)$$

$$\nabla_\theta\hat{\mathbf{r}} = \frac{1}{r}\hat{\boldsymbol{\theta}} \quad (34)$$

$$\nabla_z\hat{\mathbf{r}} = \mathbf{0} \quad (35)$$

$$\nabla_r\hat{\boldsymbol{\theta}} = \mathbf{0} \quad (36)$$

$$\nabla_\theta\hat{\boldsymbol{\theta}} = -\frac{1}{r}\hat{\mathbf{r}} \quad (37)$$

$$\nabla_z\hat{\boldsymbol{\theta}} = \mathbf{0} \quad (38)$$

$$\nabla_r\hat{\mathbf{z}} = \mathbf{0} \quad (39)$$

$$\nabla_\theta\hat{\mathbf{z}} = \mathbf{0} \quad (40)$$

$$\nabla_z\hat{\mathbf{z}} = \mathbf{0}. \quad (41)$$

The Christoffel symbols of the second kind in the definition of Misner *et al*. (1973, p. 209) are given by

$$\Gamma^r = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\frac{1}{r} & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (42)$$

$$\Gamma^\theta = \begin{bmatrix} 0 & \frac{1}{r} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (43)$$

$$\Gamma^z = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \quad (44)$$

The Christoffel symbols of the second kind in the definition of Arfken (1985) are given by

$$\Gamma^r = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -r & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (45)$$

$$\Gamma^{\theta}=\begin{bmatrix}0 & \frac{1}{r} & 0\\ \frac{1}{r} & 0 & 0\\ 0 & 0 & 0\end{bmatrix} \quad (46)$$

$$\Gamma^{z}=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix} \quad (47)$$

(Walton 1967; Arfken 1985, p. 164, Ex. 3.8.10; Moon and Spencer 1988, p. 12a).

The covariant derivatives are then given by

$$A_{j;k}=\frac{1}{g^{kk}}\frac{\partial A_j}{\partial x_k}-\Gamma^{i}_{jk}A_i, \quad (48)$$

are

$$A_{r;r}=\frac{\partial A_r}{\partial r} \quad (49)$$

$$A_{r;\theta}=\frac{1}{r}\frac{\partial A_r}{\partial \theta}-\frac{A_\theta}{r} \quad (50)$$

$$A_{r;z}=\frac{\partial A_r}{\partial z} \quad (51)$$

$$A_{\theta;r}=\frac{\partial A_\theta}{\partial r} \quad (52)$$

$$A_{\theta;\theta}=\frac{1}{r}\frac{\partial A_\theta}{\partial \theta}+\frac{A_r}{r} \quad (53)$$

$$A_{\theta;z}=\frac{\partial A_\theta}{\partial z} \quad (54)$$

$$A_{z;r}=\frac{\partial A_z}{\partial r} \quad (55)$$

$$A_{z;\theta}=\frac{1}{r}\frac{\partial A_z}{\partial \theta} \quad (56)$$

$$A_{z;z}=\frac{\partial A_z}{\partial z}. \quad (57)$$

Cross products of the coordinate axes are

$$\hat{\mathbf{r}}\times\hat{\mathbf{z}}=-\hat{\boldsymbol{\theta}} \quad (58)$$

$$\hat{\boldsymbol{\theta}}\times\hat{\mathbf{z}}=\hat{\mathbf{r}} \quad (59)$$

$$\hat{\mathbf{r}}\times\hat{\boldsymbol{\theta}}=\hat{\mathbf{z}}. \quad (60)$$

The commutation coefficients are given by

$$c_{\alpha\beta}^{\mu}\,\vec{e}_\mu=\left[\vec{e}_\alpha,\vec{e}_\beta\right]=\nabla_\alpha\vec{e}_\beta-\nabla_\beta\vec{e}_\alpha, \quad (61)$$

But

$$\left[\hat{\mathbf{r}},\hat{\mathbf{r}}\right]=\left[\hat{\boldsymbol{\theta}},\hat{\boldsymbol{\theta}}\right]=\left[\hat{\boldsymbol{\phi}},\hat{\boldsymbol{\phi}}\right]=\mathbf{0}, \quad (62)$$

so $c_{rr}^{\alpha}=c_{\theta\theta}^{\alpha}=c_{\phi\phi}^{\alpha}=0$, where $\alpha=r,\theta,\phi$. Also

$$\left[\hat{\mathbf{r}},\hat{\boldsymbol{\theta}}\right]=-\left[\hat{\boldsymbol{\theta}},\hat{\mathbf{r}}\right]=\nabla_r\hat{\boldsymbol{\theta}}-\nabla_\theta\hat{\mathbf{r}}=0-\frac{1}{r}\hat{\boldsymbol{\theta}}=-\frac{1}{r}\hat{\boldsymbol{\theta}}, \quad (63)$$

so $c_{r\theta}^{\theta}=-c_{\theta r}^{\theta}=-\frac{1}{r}$, $c_{r\theta}^{r}=c_{r\theta}^{\phi}=0$. Finally,

$$\left[\hat{\mathbf{r}},\hat{\boldsymbol{\phi}}\right]=\left[\hat{\boldsymbol{\theta}},\hat{\boldsymbol{\phi}}\right]=0. \quad (64)$$

Summarizing,

$$c^{r}=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix} \quad (65)$$

$$c^{\theta}=\begin{bmatrix}0 & -\frac{1}{r} & 0\\ \frac{1}{r} & 0 & 0\\ 0 & 0 & 0\end{bmatrix} \quad (66)$$

$$c^{\phi}=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}. \quad (67)$$

Time derivatives of the vector are

$$\dot{\mathbf{r}}=\begin{bmatrix}\cos\theta\,\dot{r}-r\sin\theta\,\dot{\theta}\\ \sin\theta\,\dot{r}+r\cos\theta\,\dot{\theta}\\ \dot{z}\end{bmatrix}=\dot{r}\,\hat{\mathbf{r}}+r\,\dot{\theta}\,\hat{\boldsymbol{\theta}}+\dot{z}\,\hat{\mathbf{z}} \quad (68)$$

$$\ddot{\mathbf{r}}=\begin{bmatrix}-\sin\theta\,\dot{r}\,\dot{\theta}+\cos\theta\,\ddot{r}-\sin\theta\,\dot{r}\,\dot{\theta}-r\cos\theta\,\dot{\theta}^2-r\sin\theta\,\ddot{\theta}\\ \cos\theta\,\dot{r}\,\dot{\theta}+\sin\theta\,\ddot{r}+\cos\theta\,\dot{r}\,\dot{\theta}-r\sin\theta\,\dot{\theta}^2+r\cos\theta\,\ddot{\theta}\\ \ddot{z}\end{bmatrix} \quad (69)$$

$$\tag{70}$$

$$= \begin{bmatrix} -2\sin\theta\,\dot r\,\dot\theta + \cos\theta\,\ddot r - r\cos\theta\,\dot\theta^2 - r\sin\theta\,\ddot\theta \\ 2\cos\theta\,\dot r\,\dot\theta + \sin\theta\,\ddot r - r\sin\theta\,\dot\theta^2 + r\cos\theta\,\ddot\theta \\ \ddot z \end{bmatrix} \tag{71}$$

$$= \left(\ddot r - r\,\dot\theta^2\right)\hat{\mathbf{r}} + \left(2\,\dot r\,\dot\theta + r\,\ddot\theta\right)\hat{\boldsymbol{\theta}} + \ddot z\,\hat{\mathbf{z}}. \tag{72}$$

Speed is given by

$$v \equiv |\dot{\mathbf{r}}| \tag{73}$$

$$= \sqrt{\dot r^2 + r^2\,\dot\theta^2 + \dot z^2}\,. \tag{74}$$

Time derivatives of the unit vectors are

$$\dot{\hat{\mathbf{r}}} = \begin{bmatrix} -\sin\theta\,\dot\theta \\ \cos\theta\,\dot\theta \\ 0 \end{bmatrix} \tag{75}$$

$$= \dot\theta\,\hat{\boldsymbol{\theta}} \tag{76}$$

$$\dot{\hat{\boldsymbol{\theta}}} = \begin{bmatrix} -\cos\theta\,\dot\theta \\ -\sin\theta\,\dot\theta \\ 0 \end{bmatrix} \tag{77}$$

$$= -\dot\theta\,\hat{\mathbf{r}} \tag{78}$$

$$\dot{\hat{\mathbf{z}}} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{79}$$

$$= \mathbf{0}. \tag{80}$$

The convective derivative is

$$\frac{D\,\dot{\mathbf{r}}}{D\,t} \equiv \left(\frac{\partial}{\partial t} + \dot{\mathbf{r}}\cdot\nabla\right)\dot{\mathbf{r}} \tag{81}$$

$$= \frac{\partial\dot{\mathbf{r}}}{\partial t} + \dot{\mathbf{r}}\cdot\nabla\dot{\mathbf{r}}. \tag{82}$$

To rewrite this, use the identity

$$\nabla(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) + (\mathbf{A}\cdot\nabla)\,\mathbf{B} + (\mathbf{B}\cdot\nabla)\,\mathbf{A} \tag{83}$$

and set $\mathbf{A} = \mathbf{B}$, to obtain

$$\nabla(\mathbf{A}\cdot\mathbf{A}) = 2\,\mathbf{A}\times(\nabla\times\mathbf{A}) + 2\,(\mathbf{A}\cdot\nabla)\,\mathbf{A}, \tag{84}$$

so

$$(\mathbf{A}\cdot\nabla)\,\mathbf{A} = \nabla\left(\tfrac{1}{2}\,\mathbf{A}^2\right) - \mathbf{A}\times(\nabla\times\mathbf{A}). \tag{85}$$

Then

$$\frac{D\,\dot{\mathbf{r}}}{D\,t} = \ddot{\mathbf{r}} + \nabla\left(\tfrac{1}{2}\,\dot{\mathbf{r}}^2\right) - \dot{\mathbf{r}}\times(\nabla\times\dot{\mathbf{r}}) \tag{86}$$

$$= \ddot{\mathbf{r}} + (\nabla\times\dot{\mathbf{r}})\times\dot{\mathbf{r}} + \nabla\left(\tfrac{1}{2}\,\dot{\mathbf{r}}^2\right). \tag{87}$$

The curl in the above expression gives

$$\nabla\times\dot{\mathbf{r}} = \frac{1}{r}\,\frac{\partial}{\partial r}\left(r^2\,\dot\theta\right)\hat{\mathbf{z}} \tag{88}$$

$$= 2\,\dot\theta\,\hat{\mathbf{z}}, \tag{89}$$

so

$$-\dot{\mathbf{r}}\times(\nabla\times\dot{\mathbf{r}}) = -2\,\dot\theta\left(\dot r\,\hat{\mathbf{r}}\times\hat{\mathbf{z}} + r\,\dot\theta\,\hat{\boldsymbol{\theta}}\times\hat{\mathbf{z}}\right) \tag{90}$$

$$= -2\,\dot\theta\left(-\dot r\,\hat{\boldsymbol{\theta}} + r\,\dot\theta\,\hat{\mathbf{r}}\right) \tag{91}$$

$$= 2\,\dot r\,\dot\theta\,\hat{\boldsymbol{\theta}} - 2\,r\,\dot\theta^2\,\hat{\mathbf{r}}. \tag{92}$$

We expect the gradient term to vanish since speed does not depend on position. Check this using the identity $\nabla\left(f^2\right) = 2\,f\,\nabla f$,

$$\nabla\left(\tfrac{1}{2}\,\dot{\mathbf{r}}^2\right) = \tfrac{1}{2}\,\nabla\left(\dot r^2 + r^2\,\dot\theta^2 + \dot z^2\right) \tag{93}$$

$$= \dot r\,\nabla\dot r + r\,\dot\theta\,\nabla\left(r\,\dot\theta\right) + \dot z\,\nabla\dot z. \tag{94}$$

Examining this term by term,

$$\dot r\,\nabla\dot r = \dot r\,\frac{\partial}{\partial t}\nabla r \tag{95}$$

$$= \dot r\,\frac{\partial}{\partial t}\hat{\mathbf{r}} \tag{96}$$

$$= \dot r\,\dot{\hat{\mathbf{r}}} \tag{97}$$

$$= \dot r\,\dot\theta\,\hat{\boldsymbol{\theta}} \tag{98}$$

$$r\,\dot\theta\,\nabla\left(r\,\dot\theta\right) = r\,\dot\theta\left[r\,\frac{\partial}{\partial t}\nabla\theta + \dot\theta\,\nabla r\right] \tag{99}$$

$$= r\,\dot\theta\left[r\,\frac{\partial}{\partial t}\left(\frac{1}{r}\,\hat{\boldsymbol{\theta}}\right) + \dot\theta\,\hat{\mathbf{r}}\right] \tag{100}$$

$$= r\,\dot\theta\left[r\left(-\frac{1}{r^2}\,\dot r\,\hat{\boldsymbol{\theta}} + \frac{1}{r}\,\dot{\hat{\boldsymbol{\theta}}}\right) + \dot\theta\,\hat{\mathbf{r}}\right] \tag{101}$$

$$= -\dot{\theta}\,\dot{r}\,\hat{\boldsymbol{\theta}} + r\,\dot{\theta}\left(-\dot{\theta}\,\hat{\mathbf{r}}\right) + r\,\dot{\theta}^2\,\hat{\mathbf{r}} \tag{102}$$

$$= -\dot{\theta}\,\dot{r}\,\hat{\boldsymbol{\theta}} \tag{103}$$

$$\dot{z}\,\nabla\dot{z} = \dot{z}\,\frac{\partial}{\partial t}\nabla z \tag{104}$$

$$= \dot{z}\,\frac{\partial}{\partial t}\hat{\mathbf{z}} \tag{105}$$

$$= \dot{z}\,\dot{\hat{\mathbf{z}}} \tag{106}$$

$$= \mathbf{0}, \tag{107}$$

so, as expected,

$$\nabla\left(\tfrac{1}{2}\,\dot{\mathbf{r}}^2\right) = \mathbf{0}. \tag{108}$$

We have already computed $\ddot{\mathbf{r}}$, so combining all three pieces gives

$$\frac{D\,\dot{\mathbf{r}}}{D\,t} = \left(\ddot{r} - r\,\dot{\theta}^2 - 2\,r\,\dot{\theta}^2\right)\hat{\mathbf{r}} + \left(2\,\dot{r}\,\dot{\theta} + 2\,\dot{r}\,\dot{\theta} + r\,\ddot{\theta}\right)\hat{\boldsymbol{\theta}} + \ddot{z}\,\hat{\mathbf{z}} \tag{109}$$

(110)

$$= \left(\ddot{r} - 3\,r\,\dot{\theta}^2\right)\hat{\mathbf{r}} + \left(4\,\dot{r}\,\dot{\theta} + r\,\ddot{\theta}\right)\hat{\boldsymbol{\theta}} + \ddot{z}\,\hat{\mathbf{z}}. \tag{111}$$

The divergence is

$$\nabla\cdot A = A^r_{;r} = A^r_{,r} + \left(\Gamma^r_{rr}\,A^r + \Gamma^r_{\theta r}\,A^\theta + \Gamma^r_{zr}\,A^z\right) + A^\theta_{,\theta} + \left(\Gamma^\theta_{r\theta}\,A^r + \Gamma^\theta_{\theta\theta}\,A^\theta + \Gamma^\theta_{z\theta}\,A^z\right) + A^z_{,z} + \left(\Gamma^z_{rz}\,A^r + \Gamma^z_{\theta z}\,A^\theta + \Gamma^z_{zz}\,A^z\right) \tag{112}$$

$$= A^r_{,r} + A^\theta_{,\theta} + A^z_{,z} + (0+0+0) + \left(\frac{1}{r} + 0 + 0\right) + (0+0+0) \tag{113}$$

$$= \frac{1}{g_r}\,\frac{\partial}{\partial r}A^r + \frac{1}{g_\theta}\,\frac{\partial}{\partial\theta}A^\theta + \frac{1}{g_z}\,\frac{\partial}{\partial z}A^z + \frac{1}{r}\,A^r \tag{114}$$

$$= \left(\frac{\partial}{\partial r} + \frac{1}{r}\right)A^r + \frac{1}{r}\,\frac{\partial}{\partial\theta}A^\theta + \frac{\partial}{\partial z}A^z, \tag{115}$$

or, in vector notation

$$\nabla\cdot\mathbf{F} = \frac{1}{r}\,\frac{\partial}{\partial r}(r\,F_r) + \frac{1}{r}\,\frac{\partial F_\theta}{\partial\theta} + \frac{\partial F_z}{\partial z}. \tag{116}$$

The curl is

$$\nabla\times\mathbf{F} = \left(\frac{1}{r}\,\frac{\partial F_z}{\partial\theta} - \frac{\partial F_\theta}{\partial z}\right)\hat{\mathbf{r}} + \left(\frac{\partial F_r}{\partial z} - \frac{\partial F_z}{\partial r}\right)\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r\,F_\theta) - \frac{\partial F_r}{\partial\theta}\right]\hat{\mathbf{z}}. \tag{117}$$

The scalar Laplacian is

$$\nabla^2 f = \frac{1}{r}\,\frac{\partial}{\partial r}\left(r\,\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\,\frac{\partial^2 f}{\partial\theta^2} + \frac{\partial^2 f}{\partial z^2} \tag{118}$$

$$= \frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\,\frac{\partial f}{\partial r} + \frac{1}{r^2}\,\frac{\partial^2 f}{\partial\theta^2} + \frac{\partial^2 f}{\partial z^2}. \tag{119}$$

The vector Laplacian is

$$\nabla^2\mathbf{v} = \begin{bmatrix} \frac{\partial^2 v_r}{\partial r^2} + \frac{1}{r^2}\,\frac{\partial^2 v_r}{\partial\theta^2} + \frac{\partial^2 v_r}{\partial z^2} + \frac{1}{r}\,\frac{\partial v_r}{\partial r} - \frac{2}{r^2}\,\frac{\partial v_\theta}{\partial\theta} - \frac{v_r}{r^2} \\ \frac{\partial^2 v_\theta}{\partial r^2} + \frac{1}{r^2}\,\frac{\partial^2 v_\theta}{\partial\theta^2} + \frac{\partial^2 v_\theta}{\partial z^2} + \frac{1}{r}\,\frac{\partial v_\theta}{\partial r} + \frac{2}{r^2}\,\frac{\partial v_r}{\partial\theta} - \frac{v_\theta}{r^2} \\ \frac{\partial^2 v_z}{\partial r^2} + \frac{1}{r^2}\,\frac{\partial^2 v_z}{\partial\theta^2} + \frac{\partial^2 v_z}{\partial z^2} + \frac{1}{r}\,\frac{\partial v_z}{\partial r} \end{bmatrix}. \tag{120}$$

The Helmholtz differential equation is separable in cylindrical coordinates and has Stäckel determinant $S = 1$ (for r, θ, z) or $S = 1/\left(1 - \xi_2^2\right)$ (for Morse and Feshbach's ξ_1, ξ_2, and ξ_3).

SEE ALSO: Cartesian Coordinates, Elliptic Cylindrical Coordinates, Helmholtz Differential Equation--Circular Cylindrical Coordinates, Polar Coordinates, Spherical Coordinates

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